Stat 610 Homework 6

Due Thursday, November 2, 11:59pm

Assignment

The Cauchy location family with location family θ has probability density function

$$f(x;\theta) = \frac{1}{\pi}\left[\frac{1}{1+(x-\theta)^2}\right]$$

In this assignment you'll look at ways of estimating θ.

1. Suppose we have n data points, $x_1, \dots, x_n$. Assuming x_i are drawn iid from a Cauchy distribution with location parameter θ, write down the log likelihood of $x_1, \dots, x_n$.
2. Give an expression for the derivative of the log likelihood with respect to the location parameter.
3. Give an expression for the second derivative of the log likelihood with respect to the location parameter.
4. Write a function that computes a maximum likelihood estimate of the location parameter using Newton's method. Your function should take as arguments:
   - The data,
   - A starting value for the location parameter,
   - A stopping criterion.

   and should return a value $\hat{\theta}$ of the maximum likelihood estimate.
5. Use your function to compute the maximum likelihood given the following ten data points: $-2.09, -2.68, -1.92, -1.76, -2.12, 2.21, 1.97, 1.61, 1.99, 2.18$. Report your results when you start θ at $-2, -1, 0, 1, 2$.
6. One-step estimation: One-step estimation is an estimation strategy in which we start off with an initial estimate of a parameter and take just one Newton step instead of trying to reach the true maximum. Write a new function, analogous to the one in part 4, that performs one-step estimation in the Cauchy location family by starting at the median of $x_1, \dots, x_n$, and taking one Newton step. The function should take just one argument, the data, and should return the estimated location parameter.
7. Estimator efficiency: Investigate the relative efficiencies of the one-step estimator and the maximum likelihood estimator. Modify the function you wrote in part 4 so that it uses the median as the starting value for the location parameter.

   For $n = 10, 100, 1000$,

- Draw n samples from a Cauchy distribution with location parameter 0 and scale parameter 1 (using `rcauchy`).
- Compute the maximum likelihood estimate on your simulated samples using the function you defined for this problem.
- Compute the one-step estimate on your simulated samples using the function you defined in part 6.

Repeat this procedure many times (100+) for each value of n, and report the variance of the one-step estimator and the maximum likelihood estimator for each value of n.

Submission parameters

Submit two files:

- A pdf giving your answers to the questions.
- An R script with the functions and other code you used.